

# Contrast in Greyscales of Graphs

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## Abstract

A *greyscale*  $f$  of a graph  $G(V, E)$  is a mapping from  $V$  to the interval  $[0, 1]$  such that  $\{0, 1\} \subseteq \text{Im}(f)$ . This function  $f$  induces another mapping  $\hat{f}$  on  $E$  by assigning to each edge the non-negative difference of the values of  $f$  on its vertices. The *contrast vector*  $\text{cont}(G, f)$  is defined as the vector  $(\hat{f}(e_1), \hat{f}(e_2), \dots, \hat{f}(e_m))$  for all edges  $e_i$  of  $G$  in such a way that  $\hat{f}(e_i) \leq \hat{f}(e_{i+1})$  for  $i = 1, 2, \dots, m-1$ . The concept of *maximum contrast vector* is presented by using the lexicographical ordering in the set of contrast vectors of all possible greyscales defined on  $G$  and a greyscale that gives rise to a maximum contrast vector is named *maximum contrast greyscale*.

The relation between finding the maximum contrast vector for the graph  $G$  and the chromatic number of  $G$  is established. Thus the maximum contrast problem is an NP-complete problem. However, the set of values of any maximum contrast greyscale for any graph is bounded by a finite set which is given. Several methods to compute the maximum contrast vector with some restrictions in are collected in this paper.

**Keywords:** graph colouring, greyscale, maximum contrast, NP-completeness.

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## 1 Introduction

Graph colouring is one of the most studied problems of combinatorial optimization because it has a wide variety of applications such as wiring of printed circuits [7, 15], resource allocation [23, 24], frequency assignment problem [1, 12, 16], a wide variety of scheduling problems [8, 11, 19, 21] or computer register allocation [2, 6, 22].

A variety of combinatorial optimization problems on graphs can be formulated similarly in the following way. Given a graph  $G(V, E)$ , a mapping  $f : V \rightarrow \mathbb{Z}$  is defined and it induces a new mapping  $\hat{f} : E \rightarrow \mathbb{Z}$  by  $\hat{f}(e) = |f(u) - f(v)|$  for

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every  $e = \{u, v\} \in E$ . Then an optimization problem is formulated from several key elements: mappings  $f$  belonging to a subset  $S$ , the image of  $V$  by  $f$  and the image of  $E$  by  $\widehat{f}$ . In particular, the classic graph colouring problem, that is, colouring the vertices of  $G$  with as few colours as possible so that adjacent vertices always have different colours, can be stated in these terms as follows:

$$\chi(G) = \min_{f \in S} |f(V)| \quad \text{where } S = \{f : V \rightarrow \mathbb{Z} \text{ such that } 0 \notin \widehat{f}(E)\}.$$

It is well known that this minimum number  $\chi(G)$  is called the chromatic number of the graph  $G$  and that its computing is an NP-hard problem [17].

It must be stressed that the classic graph colouring problem takes into account the number of colours used but not what they are. However, there are several works related to map colouring for which the nature of the colours is essential, whereas the number of them is fixed. The *maximum differential graph colouring problem* [14], or equivalently the *antibandwidth problem* [18], colours the vertices of the graph in order to maximize the smallest colour difference between adjacent vertices and using all the colours  $1, 2, \dots, |V|$ . Under the above formulation, these problems are posed as follows:

$$\max_{f \in S} \min \widehat{f}(E) \quad \text{for } S = \{f : V \rightarrow \mathbb{Z} \text{ such that } f(V) = \{1, 2, \dots, |V|\}\},$$

and therefore the complementary optimization case, the *bandwidth problem*, is given by

$$\min_{f \in S} \max \widehat{f}(E) \quad \text{for } S = \{f : V \rightarrow \mathbb{Z} \text{ such that } f(V) = \{1, 2, \dots, |V|\}\}.$$

Note that these problems are concerned with the optimization of a scalar function and mappings that take values within a discrete spectrum. Dillencourt et al. [9] studied a variation of the differential graph colouring problem under the assumption that all colours in the colour spectrum are available, more precisely, the space of colours is a three dimensional subset of  $\mathbb{R}^3$ . This makes the problem continuous rather than discrete since the mapping  $f$  has image in  $\mathbb{R}^3$  (see [9] for details).

Other colouring problems are included in the *examination scheduling problem* category (see for instance [3, 20, 21]). This problem consists of assigning a number of exams to a number of potential time periods or slots within the examination period taking into account that no student can take two or more exams at the same time. The graph  $G$  associated to the examination scheduling problem has a vertex for each exam and two vertices are adjacent whenever there is at least one student taking their corresponding exams. This way, the chromatic number of  $G$  provides the minimum number of slots needed to generate an examination period schedule.

In all these approaches the colouring functions are considered to be scalar mappings. In this work, we present an alternative method that can solve colouring

problems but focussing in the optimization of a vector that measures the differences of colours between adjacent vertices in the graph.

Thus, different concepts related to the colouring of a graph  $G = G(V, E)$  are accurately introduced in Section 2: the *contrast* and the *gradation* associated to a greyscale of  $G$ . Namely, given the graph  $G$ , a greyscale is a mapping that associates a value from the interval  $[0, 1]$  to each vertex  $v \in V$ . This assignment can be understood as an extension of the colouring of the vertices of  $G$  with grey tones. For the contrast problem, the objective is to maximize the minimum difference of tones of grey between extremes of any edge. For the gradation problem, the goal is to minimize the maximum difference of tones of grey between extremes of any edge. The gradation problem is widely studied in a work by the same authors of this paper [4].

By introducing this new problem based in a vectorial function which allocates grey tones in an equitable manner, we propose another way to maximize the difference of colours in the graph. More specifically, we are not interested in maximize the total amount of “contrast” (here contrast means the difference of colours between adjacent vertices) neither maximize the minimum contrast (scalar function) but maximize the vector whose components represent local contrast of adjacent vertices, in ascending order. This way, the main advantage of our proposal lies in the possibility of obtaining a local distribution of the contrast for every vertex in the graph.

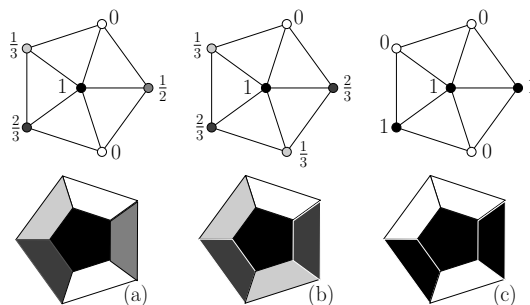


Figure 1: A comparison of three greyscales for the wheel according to their contrasts.

Figure 1 visually shows the goodness of our vectorial optimization versus a scalar optimization, based on total amount of contrast on edges or alternatively on maximin criterium.

Let us illustrate this fact by considering three colourings of the faces of the map in Figure 1. We construct its dual graph avoiding the external face in such a way that the resulting graph is the wheel. Three greyscales for the wheel are presented. Observe that according to the total amount of contrast criterium, Figure 1 (c) is an optimal solution (with total amount of contrast equal to 7 versus  $35/6$  and 5 in

Figure 1 (a) and (b), respectively). However, in (c) there are some adjacent faces having no contrast. On the other hand, (a) and (b) are solutions under the scalar maximin criterium (both of them have minimum contrast on edges equal to  $1/3$ ). Nevertheless, only (a) is an optimal solution under our vectorial criterium. In (a), it is not difficult to check that every pair of adjacent faces has the maximum possible contrast.

*Outline of the paper.* This paper is organized as follows. In Section 2 the necessary definitions about the contrast problem on graphs are established. Section 3 is devoted to the maximum contrast problem NP-completeness nature. This property is deduced from the narrow relation between the classical colouring problem and the contrast problem. Nevertheless, the set of possible values for a maximum contrast greyscale can be bounded by a set determined algorithmically as it is collected in Section 4. Also, an algorithm that calculates this set of possible values of a maximum contrast greyscale for any graph is presented and they are obtained for graphs with chromatic number up to 7. We define also restricted versions of the original problem when the grey tones 0 or 1 of some vertices are a priori known and the aim is to obtain the maximum contrast vector preserving such fixed grey tones. This problem is studied in Section 5 and it is solved for the family of complete bipartite graphs. It is also analysed in some other particular cases such as bipartite graphs and trees with some additional conditions over the set of vertices initially coloured. The last section contains a brief review of open problems and future works.

## 2 Preliminaries

This section is devoted to establishing the basic concepts about contrast on graphs and to formulating the problems to be studied in this paper. All over this paper, a graph is finite, undirected and simple and is denoted by  $G(V, E)$ , where  $V$  and  $E$  are its vertex-set and edge-set, respectively. The number of elements of  $V$  and  $E$  are denoted by  $n$  and  $m$ , respectively. Let  $N(v)$  denote the set of neighbours of the vertex  $v$  and let  $\deg(v)$  denote the *degree* of  $v$ , that is the cardinal of  $N(v)$ . For further terminology we follow [13].

Given a graph  $G(V, E)$ , a *greyscale*  $f$  of  $G$  is a mapping on  $V$  to the interval  $[0, 1]$  such that values 0 and 1 belong to  $Im(f)$ . For each vertex  $v$  of  $G$ , we call  $f(v)$  the *grey tone* of  $v$ , or more generally, the *colour* of  $v$ . Notice that two adjacent vertices may have mapped the same grey tone. In particular, values 0 and 1 are called *the extreme tones* or *white and black colours*, respectively. In a natural way, the notion of *complementary greyscale* arises for each greyscale  $f$  such that it maps every vertex  $v$  of  $G$  to  $1 - f(v)$ .

Associated to each greyscale  $f$  of the graph  $G(V, E)$ , the mapping  $\hat{f}$  is defined on  $E$  to the interval  $[0, 1]$  as  $\hat{f}(e) = |f(u) - f(v)|$  for each  $e = \{u, v\} \in E$  and it

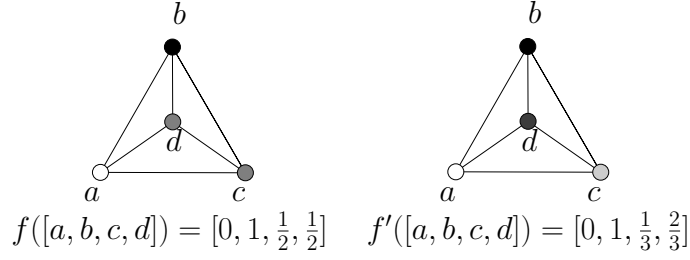


Figure 2: Two greyscales  $f$  and  $f'$  of the graph  $K_4$ .

represents the gap or increase between the grey tones of vertices  $u$  and  $v$ . The value  $\widehat{f}(e)$  is also said to be the *grey tone* of the edge  $e$ . Thus, we deal with *coloured vertices* and *edges* by  $f$  and  $\widehat{f}$ , respectively. Notice that, for any greyscale  $f$ , the same mapping  $\widehat{f}$  associated to  $f$  and its complementary one are obtained.

The *contrast vector* associated to the greyscale  $f$  of  $G$  is defined to be the vector  $\text{cont}(G, f) = (\widehat{f}(e_1), \widehat{f}(e_2), \dots, \widehat{f}(e_m))$  where the edges of  $G$  are indexed in such a way that  $\widehat{f}(e_i) \leq \widehat{f}(e_j)$  for  $i < j$ , that is, in ascending order of their grey tones. For the sake of clarity and when the graph is fixed, it can be denoted  $\text{cont}(G, f) = \mathcal{C}_f$ . Figure 2 shows two greyscales of the graph  $K_4$ ,  $f$  and  $f'$ , whose corresponding contrast vectors are  $\mathcal{C}_f = (0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$  and  $\mathcal{C}_{f'} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 1)$ , respectively.

Given two greyscales  $f$  and  $f'$  of a graph  $G$ , we say that  $f$  has *better contrast* than  $f'$  if their corresponding contrast vectors verify  $\mathcal{C}_f > \mathcal{C}_{f'}$ , following the lexicographical order. Thus, the ascending order of contrast vectors determines the goodness in terms of contrast. Then,  $f$  is said to be *smaller or greater by contrast* than  $f'$  if  $\mathcal{C}_f < \mathcal{C}_{f'}$  or  $\mathcal{C}_f > \mathcal{C}_{f'}$ , respectively. In Figure 2, the greyscale  $f'$  has better contrast than  $f$ . The *maximum contrast vector* is defined as  $\text{cont}_{\max}(G) = \max\{\text{cont}(G, f) \text{ such that } f \text{ is a greyscale of } G\}$ . If  $f$  is a greyscale of  $G$  which gives rise to the vector  $\text{cont}_{\max}(G)$ , we will say that  $f$  is a *maximum contrast greyscale* of  $G$  and the first component of  $\text{cont}_{\max}(G)$  will be called the *lightest tone* of  $G$  and denoted  $lt_G$ .

In a similar way, the *gradation vector* associated to the greyscale  $f$  of  $G$  is the vector defined as  $\text{grad}(G, f) = \mathcal{G}_f = (\widehat{f}(e_m), \widehat{f}(e_{m-1}), \dots, \widehat{f}(e_1))$ . The components of any gradation vector are ordered in decreasing order and  $f$  has *better gradation* than  $f'$  if their corresponding gradation vectors verify  $\mathcal{G}_f < \mathcal{G}_{f'}$ , according to the lexicographical order. In Figure 2,  $\mathcal{G}_f = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0) < \mathcal{G}_{f'} = (1, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ , hence the greyscale  $f$  has better gradation than  $f'$ . See [4] for more details.

The following problem arises naturally in the context of contrast. It is posed for connected graphs but general graphs can be also considered and analogous results

hold when working with each one of their connected components.

**Maximum contrast on graphs (MACG)**

*Instance:* Connected graph  $G(V, E)$ .

*Question:* Is it possible to find a greyscale of  $G$  such that its contrast vector is maximum?

We deal with the restricted version of this problem, namely when the grey tones of some vertices are a priori known and the aim is to obtain the maximum contrast vector preserving these fixed grey tones. We focuss on a particular case of this problem, namely when only white and black tones are fixed.

Given a graph  $G(V, E)$  and a nonempty proper subset  $V_c$  of  $V$ , an *incomplete  $V_c$ -greyscale* of  $G$  is a mapping on  $V_c$  to the interval  $[0, 1]$ . The vertices of  $V_c$  are named *initially coloured vertices*. A greyscale  $f$  is *compatible* with an incomplete  $V_c$ -greyscale  $g$  if  $f(u) = g(u)$  for all  $u \in V_c$ . The process of obtaining such an  $f$  is called *extending the incomplete greyscale  $g$* . The problem is established as follows and it will be formally studied in Section 5.

**$\{0, 1\}$ -Restricted maximum contrast on graphs ( $\{0, 1\}$ -RMACG)**

*Instance:* Connected graph  $G(V, E)$  and an incomplete  $V_c$ -greyscale  $g$ , where  $V_c = V_0 \cup V_1 \subset V$  with  $V_0$  and  $V_1$  disjoint subsets and such that  $g(v) = 0$  for  $v \in V_0$  and  $g(v) = 1$  for  $v \in V_1$ .

*Question:* Is it possible to find a greyscale  $f$  of  $G$  compatible with  $g$  such that its contrast vector is maximum in the set of contrast vectors of all possible greyscales compatible with  $g$  defined on  $G$ ?

If a greyscale  $f$  provides an affirmative answer to the above problem we say that  $f$  is a *maximum contrast greyscale for the  $\{0, 1\}$ -RMACG problem*. Note that  $f$  is a maximum contrast greyscale compatible with the greyscale given in the instance. The contrast vector associated to  $f$  is named *the maximum contrast vector for the  $\{0, 1\}$ -RMACG problem*.

### 3 Maximum contrast problem

In this section the problem of finding out the maximum contrast of a graph, denoted MACG, is studied. Given a graph  $G(V, E)$ , this problem consists on knowing whether a greyscale whose contrast vector is maximum can be found for  $G$ . A relation between the chromatic number and the lightest tone of the graph  $G$  is shown. The main consequence of this property is the NP-completeness of the problem MACG.

Let  $G(V, E)$  be a connected graph and let  $f$  be a greyscale of  $G$ . Our purpose is to obtain the maximum contrast vector, namely  $cont_{max}(G) = max\{cont(G, f) \mid f \text{ is a greyscale of } G\}$ . Therefore, answering the question proposed in

MACG can be considered as a maximin problem. According to the definition of better contrast, given in Section 2, it is clear that the maximum contrast vector  $cont_{max}(G)$  has no component equal to 0. Also, it is immediately deduced that a necessary condition for a greyscale  $f$  to be a maximum contrast greyscale of  $G$  is that for any vertex  $v$  with degree 1,  $f(v)$  is an extreme tone.

Given a greyscale  $f$  of  $G$ , an *incremental path of length  $k \in N$*  for  $f$  is defined to be a path of  $G$ ,  $P_k = \{u_0, e_1, u_1, e_2, u_2, \dots, e_k, u_k\}$  with  $f(u_i) = \frac{i}{k}$  for  $i = 0, 1, \dots, k$ . Thus, in any incremental path of length  $k$  all edges are coloured with the grey tone  $\frac{1}{k}$ .

We demonstrate that if  $f$  is a maximum contrast greyscale of  $G$ , then the edges coloured with the lightest tone are located in some incremental path.

Next, we study the relation between the chromatic number and the vector  $cont_{max}(G)$ . First, let us consider a graph  $G$  with chromatic number  $\chi(G) = 2$ . Any 2-colouring  $f$  of  $G$  provides a greyscale with colours 0 and 1 and a contrast vector with all components equal to 1, hence this greyscale gives rise to the vector  $cont_{max}(G)$ . We observe that in this particular case, the chromatic number equals the number of grey tones in any maximum contrast greyscale of  $G$ . This property is far from the more general case of connected graphs with  $\chi(G) \geq 3$ , which is studied next.

**Lemma 3.1.** *Let  $G(V, E)$  be a connected graph and let  $f$  be a maximum contrast greyscale of  $G$ . Let  $v \in V$  be a vertex such that  $0 < f(v) < 1$ , then there exist  $u_1$  and  $u_2 \in N(v)$  satisfying both following assertions:*

1.  $f(u_1) < f(v) < f(u_2)$ .
2.  $\widehat{f}(\{u_1, v\}) = \widehat{f}(\{u_2, v\}) = \min\{\widehat{f}(\{u, v\}) : u \in N(v)\}$ .

*Proof.* Let us consider  $a = \min\{\widehat{f}(\{u, v\}) : u \in N(v)\}$ . Since  $f$  is a maximum contrast greyscale, then  $a > 0$  trivially and since  $f(v) < 1$ , it is clear that  $a < 1$ .

Let us define the set  $\mathcal{A} = \{e \in E : \widehat{f}(e) = a\}$ , then the maximum contrast vector has  $|\mathcal{A}|$  coordinates equal to  $a$ , namely  $C_f = (c_1, c_2, \dots, c_r, a, a, \dots, a, \dots)$ . In case that  $a$  is equal to the lightest tone of  $G$  then  $r = 0$ .

Let us now partition the set  $N(v) = M_v \cup N_v$ , as follows:

$$M_v = \{u \in N(v) : \widehat{f}(\{u, v\}) = a\}, \quad N_v = \{u \in N(v) : \widehat{f}(\{u, v\}) > a\}.$$

Let us define  $b = \min\{\widehat{f}(\{u, v\}) : u \in N_v\}$  if  $N_v \neq \emptyset$  and  $b = 1$  otherwise. Since  $a < b$ , let us consider  $\varepsilon > 0$ , such that  $a + 2\varepsilon < b$  and  $0 < f(v) - \varepsilon < f(v) + \varepsilon < 1$ . Two new greyscales  $f_{v+}$  and  $f_{v-}$  are now defined as follows:

$$f_{v+}(w) = \begin{cases} f(w) & \text{if } w \neq v \\ f(v) + \varepsilon & \text{if } w = v \end{cases} \quad f_{v-}(w) = \begin{cases} f(w) & \text{if } w \neq v \\ f(v) - \varepsilon & \text{if } w = v \end{cases}$$



By definition, the greyscales  $f_{v\pm}$  verify  $\widehat{f}_{v\pm}(\{w_1, w_2\}) = \widehat{f}(\{w_1, w_2\})$  for all edges  $\{w_1, w_2\}$  of  $G$  with  $w_1, w_2 \neq v$  and  $\widehat{f}_{v\pm}(\{w_1, v\}) = a \pm \varepsilon < b \pm \varepsilon = \widehat{f}_{v\pm}(\{w_2, v\})$  for all edges  $\{w_1, v\}, \{w_2, v\}$  of  $G$  with  $w_1 \in M_v$  and  $w_2 \in N_v$ , in case that  $N_v \neq \emptyset$ .

Let us prove statement 1 by reductio ad absurdum. By assuming  $f(u) \geq f(v)$  for all  $u \in M_v$ , we obtain  $\widehat{f}_{v-}(\{u, v\}) = a + \varepsilon$ . In an analogous way, if we suppose  $f(u) \leq f(v)$  for all  $u \in M_v$ , then  $\widehat{f}_{v+}(\{u, v\}) = a + \varepsilon$ . Now, by using the properties of the functions  $f_{v+}$  and  $f_{v-}$ , it is readily deduced that both contrast vectors associated to these greyscales verify

$$C_{f_{v\pm}} = (c_1, c_2, \dots, c_r, a, \dots, a, a + \varepsilon, \dots, a + \varepsilon, \dots)$$

with  $|\mathcal{A}| - |M_v|$  coordinates equal to  $a$  and at least  $|M_v|$  coordinates equal to  $a + \varepsilon$ . Obviously, the contrast vectors associated to  $f_{v+}$  and  $f_{v-}$  are better than the contrast vector associated to  $f$ , contradicting that  $f$  is a maximum contrast greyscale. Consequently, both assumptions  $f(v) \leq f(u)$  and  $f(u) \leq f(v)$  for all  $u \in M_v$  are false and there is at least one vertex  $u_1 \in M_v$  and one vertex  $u_2 \in M_v$  such that  $f(u_1) < f(v) < f(u_2)$ . Since  $u_1$  and  $u_2$  belongs to  $M_v$ , then  $\widehat{f}(\{u_1, v\}) = \widehat{f}(\{u_2, v\}) = a$  and statement 2 holds.  $\square$

From now on, the pair of vertices  $u_1$  and  $u_2$  associated to a vertex  $v$ , given by Lemma 3.1, will be named *pair of neighbours closest to  $v$* , the vertex  $u_1$  will be named *the neighbour closest to  $v$  on the left* and the vertex  $u_2$  will be named *the neighbour closest to  $v$  on the right*.

Let us remark that the existence of such pair of vertices  $u_1, u_2$  will be determinant in the searching of possible values of  $Im(f)$ , for any maximum contrast greyscale  $f$  of  $G$ . The following proposition shows a first result concerning  $Im(f)$  although this set will be studied in detail in Section 4.

**Proposition 3.2.** *Let  $G(V, E)$  be a connected graph and let  $f$  be a maximum contrast greyscale of  $G$ . If there exists a vertex  $v \in V$  such that  $0 < f(v) < 1$ , then  $lt_G \leq \frac{1}{2}$ . Moreover, if  $lt_G = \frac{1}{2}$ , then  $Im(f) = \{0, \frac{1}{2}, 1\}$ .*

*Proof.* If  $f(v)$  is not 0 nor 1 then, from Lemma 3.1, there exists a pair of neighbours closest to  $v$ , say  $u_1$  and  $u_2$ , with  $0 \leq f(u_1) < f(v) < f(u_2) \leq 1$  such that  $f(v) - f(u_1) = f(u_2) - f(v) = \min_{u \in N(v)} \widehat{f}(\{u, v\}) \geq lt_G$ . Therefore,  $1 \geq f(u_2) - f(u_1) \geq 2lt_G$  is held and then,  $lt_G \leq \frac{1}{2}$ . In the particular case that  $lt_G = \frac{1}{2}$  we obtain  $f(u_2) - f(u_1) = 1$ , which provides necessarily  $f(u_1) = 0, f(u_2) = 1$  and  $f(v) = \frac{1}{2}$ . Consequently, we reach  $Im(f) = \{0, \frac{1}{2}, 1\}$ .  $\square$

The next lemma shows that the lightest tone  $lt_G$  is a rational number  $\frac{1}{k}$ , being  $k$  a natural number. Moreover, the lightest tone is the colour of the edges of any incremental path of length  $k$ .



**Lemma 3.3.** *Let  $G(V, E)$  be a connected graph and let  $f$  be a maximum contrast greyscale of  $G$  with lightest tone  $lt_G$ . The following statements hold:*

1. *There exists a natural number  $k$  such that  $lt_G = \frac{1}{k}$ .*
2. *For every  $e \in E$  with  $\widehat{f}(e) = lt_G$  there is at least one incremental path of length  $k$  containing  $e$ .*
3. *The vector  $cont_{max}(G)$  has at least  $k$  components equal to  $lt_G$ .*
4. *The set  $I_k = \{\frac{i}{k} : i = 0, 1, \dots, k\}$  is a subset of  $Im(f)$ .*

*Proof.* Without loss of generality, we can suppose  $lt_G < 1$ , since otherwise the result is obviously true for  $k = 1$ : the incremental path of length 1 is precisely the edge  $e$ .

Let us consider an edge  $e = \{u_0, v_0\}$  such that  $\widehat{f}(e) = lt_G < 1$ . In this case, it is clear that  $\{f(u_0), f(v_0)\} \neq \{0, 1\}$  and at least one of these vertices has degree at least two since  $G$  is connected. We suppose that  $f(u_0) < f(v_0)$ , otherwise the proof is similar.

Firstly, let us consider the case  $0 < f(u_0) < f(v_0) < 1$ , Lemma 3.1 ensures the existence of a neighbour  $v_1$  closest to  $v_0$  on the right satisfying  $\widehat{f}(\{v_0, v_1\}) = lt_G$ . Hence, by applying Lemma 3.1 to the vertex  $v_1$ , it is possible to obtain a new neighbour  $v_2$  closest to  $v_1$  on the right such that  $\widehat{f}(\{v_1, v_2\}) = lt_G$  and by iterating this process, a sequence of neighbours closest on the right  $\{v_0, v_1, \dots, v_s\}$  such that  $f(v_s) = 1$  and  $s = \left\lceil \frac{1 - f(v_0)}{lt_G} \right\rceil$  is constructed. Analogous reasoning is applied on the left of  $u_0$ . In this situation, Lemma 3.1 ensures the existence of a neighbour  $u_1$  closest to  $u_0$  on the left. Hence, by iterating this procedure, it is possible to obtain a sequence of neighbours closest on the left  $\{u_r, u_{r-1}, \dots, u_0\}$  such that  $f(u_r) = 0$  and  $r = \left\lceil \frac{f(u_0)}{lt_G} \right\rceil$ .

Let  $P_k = \{u_r, \dots, u_0, v_0, v_1, \dots, v_s\}$ , with  $k = r + s + 1$ , stand for the incremental path of  $G$ . Note that  $P_k$  satisfies  $0 = f(u_r) < f(u_{i-1}) < \dots < f(u_0) < f(v_0) < \dots < f(v_s) = 1$ .

Secondly, let us consider the case of  $f(u_0) = 0$ , hence the path  $P_k$  starts with  $u_0$ , that is  $r = 0$ . Analogously, in case that  $f(v_0) = 1$  the path  $P_k$  finishes with  $v_0$ , that is  $s = 0$ .

Finally, from the above construction, the interval  $[0, 1]$  is divided into  $k$  subintervals of equal length  $lt_G$ . Hence  $lt_G = \frac{1}{k}$  with  $k \in \mathbb{N}$ . This shows assertions 1 and 2. Observe that the  $i$ th-vertex of  $P_k$  takes the grey tone  $\frac{i}{k}$  (for  $i = 0, 1, \dots, k$ ) and consequently all its edges have grey tone equal to  $lt_G$ . Therefore, statements 3 and 4 hold and the proof is finished. □

Now we can state the main results of this section.

**Theorem 3.4.** *Given a connected graph  $G(V, E)$  with lightest tone  $lt_G = \frac{1}{k}$  and  $f$  a maximum contrast greyscale, then a  $(k + 1)$ -colouring of  $G$  is obtained from  $f$ .*

*Proof.* Let us show that the following mapping  $\Phi : V \longrightarrow \{0, 1, \dots, k\}$ , is a  $(k + 1)$ -colouring of  $G$ :

$$\Phi(v) = \begin{cases} i & \text{if } \frac{i}{k} \leq f(v) < \frac{i+1}{k} \text{ for } i = 0, \dots, k-1 \\ k & \text{if } f(v) = 1 \end{cases}$$

Since Lemma 3.3 provides one incremental path of length  $k$  whose vertices take the values  $\frac{i}{k}$  for  $i = 0, 1, \dots, k$ , the chromatic classes induced by  $\Phi$  are not empty. Next we prove that any two vertices  $u, v \in V$  with  $\Phi(u) = \Phi(v)$  are not adjacent in  $G$ . Suppose on the contrary that there is an edge  $e = \{u, v\} \in E$  such that  $\Phi(u) = \Phi(v) = i$  with  $i \neq k$ . Since  $\frac{i}{k} \leq f(u) < \frac{i+1}{k}$  and  $\frac{i}{k} \leq f(v) < \frac{i+1}{k}$  then  $\widehat{f}(e) = |f(u) - f(v)| < \frac{1}{k}$  contradicting the hypothesis  $lt_G = \frac{1}{k}$ . In addition, if  $\Phi(u) = \Phi(v) = k$  then  $\widehat{f}(e) = |f(u) - f(v)| = 0$  contradicting that  $cont_{max}(G)$  has no component equal to 0. □

**Theorem 3.5.** *The lightest tone of a connected graph  $G(V, E)$  is  $lt_G = \frac{1}{\chi(G)-1}$ .*

*Proof.* The existence of a  $(k + 1)$ -colouring of  $G$  is ensured from Theorem 3.4. Consequently,  $k + 1 \geq \chi(G)$ .

In order to prove that  $k + 1 \leq \chi(G)$ , let  $\Phi : V \longrightarrow \{0, 1, \dots, r\}$  be a colouring of  $G$  with  $r + 1 = \chi(G)$  colours and we define  $f : V \longrightarrow [0, 1]$  a greyscale of  $G$  as  $f(v) = \frac{\Phi(v)}{r}$  for all  $v \in V$ . Hence,

$$\widehat{f}(e) = |f(u) - f(v)| = \left| \frac{\Phi(u)}{r} - \frac{\Phi(v)}{r} \right| = \frac{1}{r} |\Phi(u) - \Phi(v)| \geq \frac{1}{r}$$

for every  $e = \{u, v\} \in E$ . Therefore, by definition, the first coordinate of  $C_f$ , namely  $c$ , satisfies  $\frac{1}{r} \leq c$ . Then the lightest tone of the maximum contrast vector also verifies this inequality, that is  $\frac{1}{r} \leq c \leq \frac{1}{k}$  and we reach  $k + 1 \leq r + 1 = \chi(G)$ . □

It is well known that the problem of computing the chromatic number is NP-hard [10]. As a consequence of Theorem 3.5 we can conclude this section with the following fact.

**Theorem 3.6.** *The MACG problem is NP-complete.* □

## 4 Set of values of maximum contrast greyscales

In this section we give a procedure to obtain the set of all possible values of any maximum contrast greyscale, and consequently, the components of the maximum contrast vector are determined. In accordance with Section 3, for any connected graph  $G(V, E)$ , it is known that the set  $\{\frac{i}{k} : i = 0, 1, \dots, k\}$  with  $k = \frac{1}{lt_G}$  is a subset of  $Im(f)$  (see Lemma 3.3 and recall  $lt_G = \frac{1}{\chi(G)-1}$  by Theorem 3.5). It is not difficult to find a graph for which the maximum contrast greyscale  $f$  verifies  $I_k \subsetneq Im(f)$ . In Figure 1 (top left) a maximum contrast greyscale of the wheel is given:  $f([0, 1, 2, 3, 4, 5]) = [1, 0, \frac{1}{2}, 0, \frac{2}{3}, \frac{1}{3}]$ , hence  $I_3 \subsetneq Im(f)$ .

From the definition of pair of neighbours closest to a vertex and statement 2 of Lemma 3.3, we observe that for any vertex  $v$  with colour other than the extreme tones, there exists a path of length at least 2 verifying that  $v$  is an interior point of that path. Moreover, the vertices of such path are coloured with an increasing sequence of grey tones and all edges have the same grey tone. We may guess that the set  $Im(f)$  consists of some sets of values corresponding to the colours of such paths and there are certain relations between those sets. Next we study this fact in a more deeply way as follows.

We start by defining some auxiliary notions and presenting some technical results which are concerned with properties of discrete sets of numbers which may be the grey tones of some paths. We emphasize that the analysis of these properties are made independently from the notion of greyscale of graphs.

A sequence  $[y_0, y_1, \dots, y_r] \subset [0, 1]$ , with  $r \geq 2$ , is said to be an  $h$ -step chain of length  $r$  in  $[0, 1]$  if  $y_i - y_{i-1} = h$  for  $i = 1, \dots, r$ . A number  $y_i$  for  $i \neq 0, r$  is named *interior point* and  $y_0$  and  $y_r$  are called *extreme points* of the  $h$ -step chain. Let us observe that any  $h$ -step chain is characterized by its extreme points  $y_0$  and  $y_r$  and its length  $r$ , being  $h = \frac{y_r - y_0}{r}$ .

A set of numbers  $F \subset [0, 1]$  is said to be an  $h$ -minimum-step-enchained set if  $F$  verifies the following assertions:

- (1) There is an  $h$ -step chain in  $F$  whose extreme points are precisely  $\{0, 1\}$  and there is no other  $q$ -step chain in  $F$  with extreme points  $\{0, 1\}$  and  $q < h$ .
- (2) For every  $y \in F - \{0, 1\}$  there exists a  $p$ -step chain  $P$  in  $F$  with extreme points  $y_1$  and  $y_2$  and such that  $y$  is an interior point in  $P$  for some  $p \geq h$ .  
If  $p > h$  and  $y_i \notin \{0, 1\}$ , for  $i = 1$  or  $2$ , then  $y_i$  is an interior point of a  $q_i$ -step chain with  $h \leq q_i < p$ .

An  $h$ -minimum-step-enchained set  $F$  is *maximal* if it is not a proper subset of another  $h$ -minimum-step-enchained set.

Let us illustrate these notions with an example.  $F = \{0, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, 1\}$  is a  $\frac{1}{4}$ -

minimum-step-enchained set containing the  $\frac{1}{2}$ -step chain  $[0, \frac{1}{2}, 1]$ , the  $\frac{3}{8}$ -step chains  $[0, \frac{3}{8}, \frac{3}{4}]$  and  $[\frac{1}{4}, \frac{5}{8}, 1]$ , and the  $\frac{1}{4}$ -step chain  $[0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1]$ .  $F$  also contains the  $\frac{1}{8}$ -step chain  $[\frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}]$ .

**Lemma 4.1.** *Let  $F \subset [0, 1]$  be an  $h$ -minimum-step-enchained set. The following statements hold:*

1. *There exists  $k \in \mathbb{N}$ ,  $k \geq 2$ , such that  $h = \frac{1}{k}$ .*
2.  *$I_k = \{\frac{i}{k} : i = 0, \dots, k\} \subset F$ .*
3.  *$I_k$  is the only  $\frac{1}{k}$ -step chain in  $F$ .*

*Proof.* By definition, there exists an  $h$ -step chain of length  $k$  with extreme points  $\{0, 1\}$ . Since  $k \geq 2$ ,  $[0, 1]$  is divided into  $k$  subintervals of length  $h = \frac{1}{k}$ . Hence,  $I_k$  is the set of points in this  $h$ -step chain. This leads to statements 1 and 2.

Let  $C = [y_0, y_1, \dots, y_r] \subset F$  be an  $\frac{1}{k}$ -step chain and let us suppose  $y_0 \neq 0$ . Then, the second part of assertion (2) (definition of  $h$ -minimum-step-enchained set) provides another  $q$ -step chain verifying  $\frac{1}{k} \leq q < \frac{1}{k}$ , which is impossible. In an analogous way, we reach a contradiction by supposing  $y_r \neq 1$ .  $\square$

Next we design a recursive procedure that gives a maximal  $\frac{1}{k}$ -minimum-step-enchained set for each  $k \geq 2$ . This set will be denoted by  $F_k$ , later we will prove that it is unique. The procedure starts with the set  $\{0, 1\}$  and adds all possible values  $y \in (0, 1)$  that guarantee the definition of  $\frac{1}{k}$ -minimum-step-enchained set. More particularly, the method adds all possible  $p$ -step chains, with  $p \geq \frac{1}{k}$ , in such a way that its interior and extreme points verify assertion (2) for minimum step equal to  $\frac{1}{k}$ .

We define the auxiliary mapping  $S_{H,k}$  which will help us in the checking of assertion (2) during the procedure given below.

Let  $H \subset [0, 1]$  be a finite set of numbers and  $k \in \mathbb{N}$  with  $k \geq 2$ . The function  $S_{H,k} : H \longrightarrow [0, 1]$  is defined by:

$$S_{H,k}(y) = \begin{cases} \min\{p \geq \frac{1}{k} : y \text{ is an interior point for some } p\text{-step chain in } H \\ \text{verifying assertion (2)}\} \\ 0 \text{ otherwise} \end{cases}$$

The following procedure starts with the set  $H = \{0, 1\}$  and adds possible values of new  $p$ -step chains  $[y_0, \dots, y_r]$  with  $p \geq \frac{1}{k}$  recursively whenever  $\max\{S_{H,k}(y_0), S_{H,k}(y_r)\} < p$  is true. These new values are stored in the set  $H$  and  $S_{H,k}$  must be actualized in each step. The algorithm ends when it is not possible the addition of a new interior point of a chain that satisfies assertion (2) of the definition of  $h$ -minimum-step-enchained set.

PROCEDURE: MAXIMAL ENCHAINED SET (MES)

**Input:** A natural number  $k \geq 2$ .

**Output:** The maximal  $\frac{1}{k}$ -minimum-step-enchained set,  $F_k$ .

1. Initialize  $H$ ,  $New \leftarrow \{0, 1\}$
2. Initialize  $S_{H,k}(y) = 0$  for all  $y \in H$
3. **While**  $|New| > 0$  **do**
  - (a) Initialize  $New \leftarrow \emptyset$
  - (b) **For each**  $\{y_1, y_2\} \subset H$  **do**
    - i. Initialize  $r \leftarrow 2$
    - ii. Initialize  $p \leftarrow \frac{|y_2 - y_1|}{r}$
    - iii. **While**  $p \geq \frac{1}{k}$  **do**
      - A. **If**  $p > \max\{S_{H,k}(y_1), S_{H,k}(y_2)\}$  **do**
        - I. Make  $C_r$  the  $p$ -step chain with extremes  $\{y_1, y_2\}$ .
        - II. **For each**  $y \in C_r - \{y_1, y_2\}$  **do**
          - $\alpha$ . **If**  $y \notin H$  **then**  $S_{H,k}(y) = p$
          - $\beta$ . **If**  $S_{H,k}(y) > p$  **then**  $S_{H,k}(y) = p$
        - III. Actualize  $New \leftarrow (New \cup (C_r - H))$
      - B. Actualize  $r \leftarrow r + 1$
      - C. Actualize  $p \leftarrow \frac{|y_2 - y_1|}{r}$
    - (c) Actualize  $H \leftarrow (H \cup New)$
  4. **return**  $F_k = H$

Since the procedure starts with two fixed numbers, these are  $\{0, 1\}$ , and each step is deterministic, the uniqueness of its output is straightforwardly deduced. Notice that this affirmation is true whenever the finiteness of the procedure is proven (Theorem 4.3). By construction, the actualized  $H$  after the execution of Step 3.(c) of the procedure is a  $\frac{1}{k}$ -minimum-step-enchained set. The output of the algorithm is the maximal  $\frac{1}{k}$ -minimum-step-enchained set,  $F_k$ , due to the fact that every pair of possible values for  $H$  are revisited in Step 3.(b) until no new value can be adjoined to  $H$ . Moreover,  $F_k$  is a rational set due to the fact that only rational numbers are adjoined to  $C_r$  in Step 3.(b)iii.A.I of the algorithm.

The finiteness of  $F_k$  (and consequently of the procedure) deserves a detailed analysis which starts with the following technical lemma. Let us establish some

notation. First of all,  $F_k$  can be described as  $F_k = \cup_{i=0}^{\infty} A_i$  where  $A_0 = \{0, 1\}$  and, for  $i \geq 1$ ,

$$A_i = \{y \in F_k - \cup_{j=0}^{i-1} A_j : y \text{ verifies assertion (2) such that the extremes } y_1 \text{ and } y_2 \text{ of the existing } p\text{-step chain satisfy } y_1, y_2 \in \cup_{j=0}^{i-1} A_j\}.$$

For the sake of simplicity, from now on we will denote  $F_k^i = \cup_{j=0}^i A_j$ .

**Lemma 4.2.** *The following statements hold for any  $A_i$  with  $i \geq 1$ :*

1. *Every number  $y \in A_i$  is an interior point for a  $p$ -step chain in  $F_k$  with extreme points  $y_1 < y_2$  and  $\{y_1, y_2\} \cap A_{i-1} \neq \emptyset$ , where  $p = S_{F_k^i, k}(y)$ .*
2. *Symetry property: if  $y \in A_i$  then  $1 - y \in A_i$  and  $S_{F_k^i, k}(y) = S_{F_k^i, k}(1 - y)$ .*
3.  *$A_i$  is a finite set.*
4. *Let be  $p_i = \min\{S_{F_k^i, k}(y), : y \in A_i\}$ , then  $p_i > p_{i-1}$ , where  $p_0 = 0$ .*

*Proof.* Statement 1 holds by definition of the set  $A_i$ . Statement 2 is demonstrated by induction over  $i$ : it is trivially true for  $A_0 = \{0, 1\}$  and let us suppose that it is also true for  $A_j$  with  $j \leq i$ . Let us consider  $y \in A_{i+1}$ . Then,  $y$  is an interior point of a  $p$ -step chain with extreme points  $y_1 < y_2$ , such that  $y_s \in F_k^i$  and  $S_{F_k^{i+1}, k}(y) > \max\{S_{F_k^i, k}(y_1), S_{F_k^i, k}(y_2)\}$  for  $s = 1, 2$ . From the induction hypothesis,  $1 - y_s \in F_k^i$  and  $S_{F_k^i, k}(1 - y_s) = S_{F_k^i, k}(y_s)$ , for  $s = 1, 2$ . We make the  $p$ -step chain with extreme points  $1 - y_2 < 1 - y_1$ . Then,  $1 - y$  is an interior point and  $S_{F_k^{i+1}, k}(1 - y) = S_{F_k^{i+1}, k}(y) > \max\{S_{F_k^i, k}(y_1), S_{F_k^i, k}(y_2)\} = \max\{S_{F_k^i, k}(1 - y_1), S_{F_k^i, k}(1 - y_2)\}$ , therefore  $1 - y \in A_{i+1}$ .

Statement 3 is demonstrated by induction over  $i$ : it is true for  $i = 0$ , since  $|A_0| = 2$ . Let us suppose  $|A_j| < +\infty$  for  $j \leq i$ , then  $|F_k^i| = |\cup_{j=0}^i A_j| \leq |A_0| + |A_1| + \dots + |A_i| < +\infty$ . According to Step 3 (b) iii.A.I. of MES procedure, in  $A_{i+1}$  there are at most  $\binom{|F_k^i|}{2}$  chains with length 2, 3,  $\dots$   $k$ . Therefore,  $|A_{i+1}|$  has at most

$$(1 + 2 + \dots + (k - 1)) \binom{|F_k^i|}{2} = \frac{(k)(k - 1)}{2} \binom{|F_k^i|}{2}$$

elements, that is,  $|A_{i+1}| < +\infty$ .

For the proof of statement 4, let us observe that if  $y \in A_i$  then there is some  $h$ -step chain with extreme points  $y_1 < y_2$  and  $h = S_{F_k^i, k}(y) > \max\{S_{F_k^{i-1}, k}(y_1), S_{F_k^{i-1}, k}(y_2)\}$  and from statement 2,  $y_1$  or  $y_2$  belongs to  $A_{i-1}$ , hence  $S_{F_k^{i-1}, k}(y_s) \geq p_{i-1}$  for some  $s = 1, 2$ . Then,  $S_{F_k^i, k}(y) > \max\{S_{F_k^{i-1}, k}(y_1), S_{F_k^{i-1}, k}(y_2)\} \geq p_{i-1}$ , that is,  $S_{F_k^i, k}(y) > p_{i-1}$  for each  $y \in A_i$ . Since  $|A_i| < +\infty$ , it must be  $p_i = \min\{S_{F_k^i, k}(y), \text{ such that } y \in A_i\} > p_{i-1}$ .

□

**Theorem 4.3.**  $F_k$  is a finite set for every  $k \geq 2$ .

*Proof.* By reductio ad absurdum, let us suppose  $|F_k| = +\infty$ . Since  $F_k = \cup_{i=0}^{\infty} A_i$  and each set  $A_i$  is finite, then it must be  $A_i \neq \emptyset$  for all  $i$ , and hence,  $\{p_i\}_{i=0}^{+\infty}$  is an infinite strictly increasing succession of numbers with trivial upper bound  $p_i < \frac{1}{2}$ . Then, there exists  $\lim_{i \rightarrow +\infty} p_i = p$ .

On the other hand, for each  $a \in A_i$  and  $0 < a < \frac{1}{3}$ , by the symetry property, the number  $b = 1 - a > \frac{2}{3}$  lies in  $A_i$  and  $c = \frac{b}{2} = \frac{1-a}{2} \in A_{i+1}$  because  $c$  is an interior point of the  $c$ -step chain  $[0, c, b]$  with  $S_{F_k, k}(c) = c > \frac{1}{3} > S_{F_k, k}(b) = S_{F_k, k}(a)$ . Let us observe that  $(c - \frac{1}{3}) = \frac{1-a}{2} - \frac{1}{3} = \frac{\frac{1}{3}-a}{2}$ , that is, for every  $a \in A_i$  with  $a < \frac{1}{3}$  there exists  $c \in A_{i+1}$  with  $|c - \frac{1}{3}| = \frac{1}{2}|a - \frac{1}{3}|$ . Under the assumption  $F_k$  is an infinite set, then value  $\frac{1}{3}$  is an accumulation point in  $F_k$  and then,  $p = \lim_{i \rightarrow +\infty} p_i = \frac{1}{3}$ .

Now, by definition of limit, for all  $\epsilon > 0$ , there is some  $m$  such that if  $i \geq m$  then  $\frac{1}{3} - p_i < \epsilon$ . Let be  $y \in A_{m+1}$ , then  $y$  is an interior point of an  $h$ -step chain with extreme points  $y_1 < y_2$  and some  $y_s \in A_m$  for  $s = 1, 2$ , where  $h \geq p_{m+1}$ . Let us suppose  $y_2 \in A_m$ , then  $y_2$  is an interior point of a  $q$ -step chain with extreme points  $y_3 < y_4$ , where  $q \geq p_m$ . Then  $0 < y_1 < y < y_2 < y_4 < 1$  and  $y_4 - y_1 = (y_4 - y_2) + (y_2 - y) + (y - y_1) \geq p_{m+1} + p_{m+1} + p_m > 3(\frac{1}{3} - \epsilon) = 1 - 3\epsilon$ . recall that  $y_4 - y_1 \geq 1 - \frac{1}{k}$ , and then,  $1 - 3\epsilon < 1 - \frac{1}{k}$ , so  $\epsilon > \frac{1}{3k}$  which is a contradiction. Therefore, the assumption is false and the set  $F_k$  is finite for every  $k \geq 2$ .  $\square$

As a consequence of Theorem 4.3, if  $H_i$  denotes the set  $H$  after  $i$  loops in MES procedure, then the number of loops is finite because  $F_k$  is finite. This demonstrate the finiteness of the procedure. On the other hand, from the proof of statement 3 in Lemma 4.2, let us remark that  $H_i$  can be computed in  $\mathcal{O}(|H_i|)$  time complexity. This leads to an exponential time complexity in the worst case for large values of  $k$ . Nevertheless, it is worth to run the MES procedure for  $k$  from 2 to 7. This will allow us to obtain some examples of maximum contrast of graphs as it will be shown below. The following sets are obtained:

$$F_2 = \{0, \frac{1}{2}, 1\}$$

$$F_3 = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$$

$$F_4 = \{0, \frac{1}{4}, \frac{1}{3}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{2}{3}, \frac{3}{4}, 1\}$$

$$F_5 = \{0, \frac{1}{5}, \frac{1}{4}, \frac{4}{15}, \frac{3}{10}, \frac{1}{3}, \frac{7}{20}, \frac{11}{30}, \frac{3}{8}, \frac{2}{5}, \frac{7}{15}, \frac{19}{40}, \frac{1}{2}, \frac{21}{40}, \frac{8}{15}, \frac{3}{5}, \frac{5}{8}, \frac{19}{30}, \frac{13}{20}, \frac{2}{3}, \frac{7}{10}, \frac{11}{15}, \frac{3}{4}, \frac{4}{5}, 1\}$$

$$F_6 = \{0, \frac{1}{6}, \frac{1}{5}, \frac{5}{24}, \frac{2}{9}, \frac{1}{4}, \frac{7}{27}, \frac{19}{72}, \frac{4}{15}, \frac{5}{18}, \dots\}$$

$$F_7 = \{0, \frac{1}{7}, \frac{1}{6}, \frac{6}{35}, \frac{5}{28}, \frac{4}{21}, \frac{1}{5}, \frac{17}{84}, \frac{23}{112}, \frac{13}{63}, \dots\}$$

The sets  $F_6$  ad  $F_7$  are collected in a data file which is available at [5]. Table 1



shows the increase of  $F_k$  cardinality. Note that the cases for  $k \geq 6$  are particularly significant.

k	2	3	4	5	6	7
$ F_k $	3	5	9	25	145	19027

Table 1: Cardinality of some  $F_k$ .

With these technical results we have established some properties of discrete sets of numbers that will be used to demonstrate the main theorem of this section. The next theorem gives a fundamental property of the set  $Im(f)$  for any maximum contrast greyscale  $f$  of a given graph  $G$  with known chromatic number  $\chi(G)$ .

**Theorem 4.4.** *Let  $G(V, E)$  be a connected graph with chromatic number  $\chi(G) = k + 1$ . For any maximum contrast greyscale  $f$  of  $G$ ,  $Im(f)$  is a  $\frac{1}{k}$ -minimum-step-enchained set and  $I_k \subset Im(f) \subset F_k$  is verified.*

*Proof.* Let us consider an arbitrary maximum contrast greyscale  $f$  of  $G$ . Firstly, we show that  $Im(f)$  satisfies assertion (1) in the definition of  $\frac{1}{k}$ -minimum-step-enchained set. From Lemma 3.3 and Theorem 3.5 it holds that  $I_k \subset Im(f)$  is a  $\frac{1}{k}$ -step chain with extreme points  $\{0, 1\}$ , where  $k = \chi(G) - 1$ . Suppose there exists a  $q$ -step chain with extreme points  $\{0, 1\}$  and  $q < \frac{1}{k}$ , namely  $[0, q, \dots, 1]$ . Then,  $q = f(v)$  for some  $v \in V$  and, by Lemma 3.1, there exists  $u_1 \in N(v)$ , a neighbour closest to  $v$  on the left. Hence,  $f(u_1) < f(v)$  and  $\widehat{f}(\{u_1, v\}) \leq q < \frac{1}{k}$ , the lightest tone, which is a contradiction. Therefore,  $I_k$  is the only  $\frac{1}{k}$ -chain with extremes  $\{0, 1\}$  and assertion (1) holds.

In order to prove assertion (2) in the definition of  $\frac{1}{k}$ -minimum-step-enchained set we use the following auxiliary mapping  $C : V \rightarrow [0, 1]$  defined as follows:

$$C(v) = \min_{w \in f^{-1}(f(v))} \{\widehat{f}(\{u, w\}) : u \in N(w)\}$$

In other words,  $C$  computes the minimum grey tone over the set of all edges incident in vertices coloured with the grey tone  $f(v)$ .

Given a value  $y \in Im(f) - \{0, 1\}$ , let  $v$  be a vertex such that  $y = f(v)$ . Let us consider  $C(v)$  and a vertex  $w \in C^{-1}(C(v))$ . Since  $f(w) = y \notin \{0, 1\}$ , the hypotheses of Lemma 3.1 are verified and hence there is a pair of neighbours closest to  $w$ , namely  $u_1$  and  $v_1$  such that  $f(u_1) < f(w) < f(v_1)$  and  $\widehat{f}(\{u_1, w\}) = \widehat{f}(\{v_1, w\}) = p$ , where  $p = C(v)$ . Therefore,  $[f(u_1), f(w) = y, f(v_1)]$  is a  $p$ -step chain. Moreover, and since  $w \in N(u_1)$ , it is held that  $C(u_1) \leq \widehat{f}(\{u_1, w\}) = C(w) = C(v) = p$  (see Lemma 3.1); analogously  $C(v_1) \leq p$ .

Let us suppose  $C(u_1) = C(w) = p$  and  $f(u_1) \neq 0$ . A similar reasoning gives rise to  $w_2 \in C^{-1}(C(u_1))$  and there exists  $u_2$ , the neighbour closest to  $w_2$  on the left such that  $f(u_2) = f(u_1) - p$ , and the above  $p$ -step chain is enlarged on the left as follows  $[f(u_2), f(u_1), f(w) = y, f(v_1)]$ . This procedure can be repeated  $r_1$  times until  $C(u_{r_1}) < C(w) = p$  or else  $f(u_{r_1}) = 0$ . This way, a left extreme point for the  $p$ -step chain with interior point  $y$  is found. In summary, we obtain  $[f(u_{r_1}), \dots, f(u_2), f(u_1), f(w) = y, f(v_1)]$ . The case  $C(v_1) = C(w) = p$  and  $f(v_1) \neq 1$  can be tackled similarly and the  $p$ -step chain is completed on the right in the form:  $[f(u_{r_1}), \dots, f(u_2), f(u_1), f(w) = y, f(v_1), f(v_2), \dots, f(v_{r_2})]$  whose extreme points are 0 or 1 or otherwise  $C$  reaches a value less than  $p$ .

Notice that, from the definition of lightest tone of  $G$ , for every vertex  $u$ ,  $C(u) \geq \frac{1}{k}$  holds, in particular  $C(v) = p \geq \frac{1}{k}$ . Therefore, every  $y \in \text{Im}(f) - \{0, 1\}$  is an interior point of a  $p$ -step chain with  $p \geq \frac{1}{k}$  and whose extreme points are  $f(u_{r_1}) = 0$  or else  $C(f(u_{r_1})) = q_1 < p$  or  $f(v_{r_2}) = 1$  or else  $C(f(v_{r_2})) = q_2 < p$ . In such a case, a similar construction gives rise to a  $q_i$ -step chain with  $q_i < p$  and  $f(u_{r_1})$  or  $f(v_{r_2})$  are the corresponding interior points.

We conclude that  $\text{Im}(f)$  is a  $\frac{1}{k}$ -minimum-step-enchained set and from the maximality of  $F_k$  it is reached that  $\text{Im}(f) \subset F_k$  and the proof is finished.  $\square$

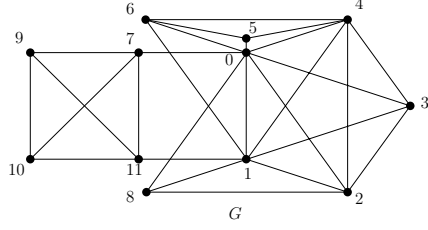
As an immediate consequence of Theorems 4.3 and 4.4 we reach the next corollary which assures that the notion of *maximum contrast vector* is well defined.

**Corollary 4.5.** *For any graph  $G$ , the set*

$$\{\text{cont}(G, f) \text{ such that } f \text{ is a greyscale of } G\}$$

*has a maximum.*

A natural question arises: we wonder if the bound set  $\text{Im}(f) \subset F_k$  for a given maximum contrast greyscale  $f$  is tight or otherwise there exists  $y \in F_k$  such that no graph  $G(V, E)$  with  $\chi(G) = k + 1$  and no maximum contrast greyscale  $f$  of  $G$  verifies  $y \in \text{Im}(f)$ . For  $k = 2$  and the complete graph  $G = K_3$ ,  $\text{Im}(f) = F_2$ . Figure 1 ((a) top) and Figure 3 show two graphs for which  $\text{Im}(f) = F_k$ , with  $k = 3, 4$ , respectively. A brute-force algorithm has been implemented for these two graphs in order to check that a maximum contrast greyscale has precisely these sets of values. Let us remark that in the latter case, the computation for  $k = 4$  requires at most  $12^9$  combinations of values to test maximum contrast greyscales. In the general case, we guess that the bound  $\text{Im}(f) \subset F_k$  is tight but constructing particular examples verifying it is a difficult task since an amount of vertices  $|V| \geq |F_k|$  is required and we are dealing with an NP-complete problem.



$$f([0, 1, \dots, 11]) = [1, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 0, \frac{3}{8}, 0, \frac{5}{8}, \frac{2}{3}, \frac{1}{3}, 1]$$

Figure 3: A maximum contrast greyscale of the graph  $G$  with  $\chi(G) = 5$  and  $Im(f) = F_4$ .

## 5 Restricted maximum contrast problem on bipartite graphs

In this section we consider the family of bipartite graphs or equivalently 2-chromatic graphs. For these graphs we deal with the version of the maximum contrast problem in which the graph has several vertices initially coloured with extreme tones 0 (white) and 1 (black) and the rest of the vertices must be coloured by preserving those initial colours. We name it  **$\{0, 1\}$ -Restricted maximum contrast on graphs** ( $\{0, 1\}$ -RMACG) and it was formally defined in Section 2.

Recall that  $V_c$  denotes the set of vertices initially coloured, in particular,  $V_i$  denotes the set of vertices initially coloured with extreme tone  $i$ , for  $i = 0, 1$ . All over this section we consider that no pair of vertices in  $V_i$  are adjacent, for  $i = 0, 1$ . Let us observe that in case that adjacent vertices are allowed to be initially coloured with the same extreme tone, then the first component of the maximum contrast vector is equal to zero. In this particular case, the solution of  $\{0, 1\}$ -RMACG problem is essentially the same as in the case in which the edges with grey tone equal to zero are removed from the graph. Hence it is worth to focus on the problem with this extra condition of no adjacent vertices having the same initial colour.

It suffices to revisit the proofs of Lemma 3.1 and Proposition 3.2 to realize that similar results hold for the  $\{0, 1\}$ -RMACG problem, which are included next.

**Lemma 5.1.** *Let  $G(V, E)$  be a connected graph and let  $f$  be a maximum contrast greyscale compatible with an incomplete  $V_c$ -greyscale  $g$ . Let  $v \in V - V_c$  be a vertex such that  $0 < f(v) < 1$ , then there exist  $u_1$  and  $u_2 \in N(v)$  satisfying both following assertions:*

1.  $f(u_1) < f(v) < f(u_2)$ .
2.  $\widehat{f}(\{u_1, v\}) = \widehat{f}(\{u_2, v\}) = \min\{\widehat{f}(\{u, v\}) : u \in N(v)\}$ .

**Proposition 5.2.** *Let  $G(V, E)$  be a connected graph and let  $f$  be a maximum contrast greyscale compatible with an incomplete  $V_c$ -greyscale  $g$ . If there exists a vertex  $v \in V$  such that  $0 < f(v) < 1$ , then the first component of  $\mathcal{C}_f$  is at most  $\frac{1}{2}$ . Moreover, if the first component of  $\mathcal{C}_f$  is equal to  $\frac{1}{2}$ , then  $Im(f) = \{0, \frac{1}{2}, 1\}$ .*

The pair of vertices  $u_1$  and  $u_2$  associated to a vertex  $v$ , given by Lemma 5.1, will be referred as the pair of neighbours closest to  $v$ , the vertex  $u_1$  is the neighbour closest to  $v$  on the left and the vertex  $u_2$  is the neighbour closest to  $v$  on the right.

From here on, unless other thing is stated, we consider the set  $V_c$  having all initially coloured vertices with extreme tones. Let us fix some notation that will help in the remaining of this paper. Any 2-chromatic graph  $G$  has precisely two 2-colourings, say  $\phi_0$  and  $\phi_1$ , both using colours 0 and 1. Let  $g$  be an incomplete  $V_c$ -greyscale of  $G$ . Let us partition the set of initially coloured vertices as follows:  $V_c = V_{\phi_0} \cup V_{\phi_1}$ , where  $V_{\phi_0}$  denotes the set of vertices of  $V_c$  whose colour coincides with the colour assigned by the colouring  $\phi_0$ . Analogously,  $V_{\phi_1}$  denotes the set of vertices of  $V_c$  whose colour coincides with the colour assigned by the colouring  $\phi_1$ , and hence their colours are opposite to the colour assigned by  $\phi_0$ . Let  $f$  be a maximum contrast greyscale for the  $\{0, 1\}$ -RMACG problem. Observe that  $V_c = V_{\phi_0}$  (equivalently  $V_c = V_{\phi_1}$ ) implies  $Im(f) = \{0, 1\}$ , that is,  $f = \phi_0$  ( $f = \phi_1$ , respectively). The case  $V_{\phi_0} \neq \emptyset$  and  $V_{\phi_1} \neq \emptyset$  will be also studied. In fact, both situations occur for certain bipartite graphs.

Next, we give an upper bound set for  $Im(f)$  being  $f$  a maximum contrast greyscale for the  $\{0, 1\}$ -RMACG problem on the family of 2-chromatic graphs and the set  $V_c$  of arbitrary cardinality.

**Theorem 5.3.** *Let  $G(V, E)$  be a 2-chromatic connected graph and let  $f$  be a maximum contrast greyscale for the  $\{0, 1\}$ -RMACG problem on  $G$ . Then, it is verified that  $Im(f) \subseteq \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$ .*

*Proof.* Let us consider the set of initially coloured vertices  $V_c \subset V$  with incomplete greyscale  $g$ . Consider the 2-colouring of  $G$ ,  $\varphi = \phi_i : V \rightarrow \{0, 1\}$ ,  $i = 0, 1$ , such that  $V_{\phi_i} \neq \emptyset$  (it is not difficult to check that at least one of the two 2-colourings  $\phi_1$  or  $\phi_2$  of  $G$  satisfies this condition). We define the following greyscale on  $G$ :

$$f_\varphi(v) = \begin{cases} g(v) & \text{if } v \in V_c \\ \frac{2}{3} & \text{if } v \notin V_c \text{ and there is } u \in V_c \cap N(v) \text{ and } \varphi(u) \neq g(u) = 0 \\ \frac{1}{3} & \text{if } v \notin V_c \text{ and there is } u \in V_c \cap N(v) \text{ and } \varphi(u) \neq g(u) = 1 \\ \text{and} & \text{there is no } u \in V_c \cap N(v) \text{ such that } \varphi(u) \neq g(u) = 0 \\ \varphi(v) & \text{otherwise.} \end{cases}$$

It is readily checked that  $f_\varphi$  is well defined and compatible with  $g$ . Besides, since  $G$  is bipartite and equivalently it has no odd cycles,  $Im(\widehat{f}_\varphi) \subseteq \{\frac{1}{3}, \frac{2}{3}, 1\}$  holds.

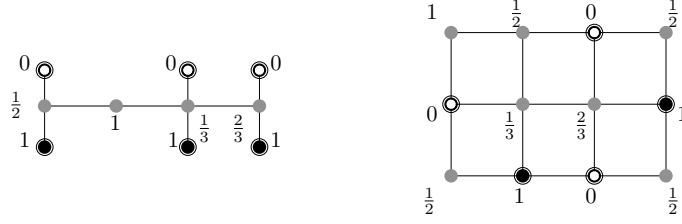


Figure 4: Maximum contrast greyscales for the  $\{0, 1\}$ -RMACG problem on this tree and this grid are given. Double circles in vertices denote the set  $V_c$  of initially coloured vertices.

Hence, the contrast vector  $\mathcal{C}_{f_\varphi}$  has first component  $\frac{1}{3}$  and, therefore, the maximum contrast vector compatible with  $g$  must have first component  $a \geq \frac{1}{3}$ .

Next let  $f$  be a maximum contrast greyscale compatible with  $g$  and let us suppose that there is a vertex  $u \in V$  such that  $f(u) \notin \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$ . From Lemma 5.1, there exists the pair of neighbours closest to  $u$ ,  $u_1$  and  $u_2$ , such that  $f(u_1) < f(u) < f(u_2)$  and  $f(u) - f(u_1) = f(u_2) - f(u) = d$ . We analyse the value  $d$ . Observe that  $d < \frac{1}{2}$ , since  $d = \frac{1}{2}$  implies  $f(u) = \frac{1}{2}$  which is ruled out. On the other hand,  $\widehat{f}(\{u, u_1\}) = d \geq a \geq \frac{1}{3}$ , thus  $\frac{1}{3} \leq d < \frac{1}{2}$ .

Since  $d < \frac{1}{2}$ , it is clear that  $f(u_1) \neq 0$  or  $f(u_2) \neq 1$ . Let us suppose  $f(u_1) \neq 0$  and consider the pair of neighbours closest to  $u_1$ , namely  $u_3$  and  $u_4$ , given by Lemma 5.1. Then,  $f(u_3) < f(u_1) < f(u_4)$  and  $\frac{1}{3} \leq f(u_1) - f(u_3) = f(u_4) - f(u_1) \leq d$ . Therefore  $f(u_2) - f(u_3) \geq \frac{1}{3} + 2d \geq 1$ , which implies  $d = \frac{1}{3}$  and  $f(u) = \frac{2}{3}$  contradicting our assumption. In a similar manner, if  $f(u_2) \neq 1$  we reach  $f(u) = \frac{1}{3}$  which is also impossible.

Then, we conclude  $Im(f) \subseteq \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$  and the proof is finished.  $\square$

The bound set given by Theorem 5.3 is tight in the sense that there are examples of graphs for which the full set  $\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$  is needed. Figure 4 shows two of such examples.

However, next theorem gives a characterization of  $Im(f)$  on the family of complete bipartite graphs, where  $f$  never takes the values  $\frac{1}{3}$  and  $\frac{2}{3}$  no matter the cardinality of  $V_c$  is.

**Theorem 5.4.** *Let  $f$  be a maximum contrast greyscale for the  $\{0, 1\}$ -RMACG problem on any complete bipartite graph. Then, its maximum contrast vector has either all components equal to 1 or else all components equal to  $\frac{1}{2}$ .*

*Proof.* Let  $f$  be a maximum contrast greyscale compatible with an incomplete  $V_c$ -greyscale  $g$  on the complete bipartite graph  $K_{r,s}$ . Since  $V_c$  has no adjacent vertices

with the same initial extreme tone, only two possibilities occur. The first one is that all vertices of  $V_c$  have assigned the colour of either  $\phi_0$  or  $\phi_1$ , that is,  $V_c = V_{\phi_0}$  or  $V_c = V_{\phi_1}$ , respectively. Without loss of generality, we consider  $V_c = V_{\phi_0}$ . In this case, precisely  $f = \phi_0$  gives an affirmative answer to the  $\{0, 1\}$ -RMACG problem on  $K_{r,s}$ . Therefore, its maximum contrast vector has all the components equal to 1. The other possibility is produced when  $V_{\phi_0} \neq \emptyset$  and  $V_{\phi_1} \neq \emptyset$ . It is deduced that  $V_c$  is a subset of one of the chromatic classes of the complete bipartite graph  $K_{r,s}$ . Thus it is straightforwardly checked that all vertices of the other chromatic class of the graph must have assigned the grey tone  $\frac{1}{2}$  in order to give a maximum contrast greyscale for the  $\{0, 1\}$ -RMACG problem on  $K_{r,s}$ . Hence, its maximum contrast vector has all the components equal to  $\frac{1}{2}$ .  $\square$

This result shows that the  $\{0, 1\}$ -RMACG problem is solved on the class of complete bipartite graphs. The general case of our problem on bipartite graphs is far from having a trivial answer. Next Theorems 5.6 and 5.8 illustrate results for particular situations. In the first one only one vertex lies in  $V_{\phi_0}$  or  $V_{\phi_1}$ , while in the second one the bipartite graph is a tree. Below we include the technical Lemmas 5.5 and 5.7 needed for proving the announced theorems. We denote  $P_{uv}$  a  $u - v$  path in  $G$ , that is a path joining the vertices  $u$  and  $v$  in  $G$  as it is usually collected in the literature (see [13]). According to the notation introduced above, the following result is given.

**Lemma 5.5.** *Let  $f$  be a maximum contrast greyscale for the  $\{0, 1\}$ -RMACG problem on a 2-chromatic connected graph  $G$ . Then, for any pair of vertices  $u \in V_{\phi_0}$ ,  $v \in V_{\phi_1}$  and for any  $u - v$  path in  $G$  there is a vertex  $w$  in this path such that  $f(w) \notin \{0, 1\}$ .*

*Proof.* Let  $P_{uv}$  be a  $u - v$  path in  $G$  with length  $l$ . Since  $G$  is a bipartite graph, for any 2-colouring  $\phi_i$  of  $G$  it is verified that either  $l$  is an even number if and only if  $\phi_i(u) = \phi_i(v)$  or else  $l$  is an odd number if and only if  $\phi_i(u) \neq \phi_i(v)$ , for  $i = 0, 1$ . Therefore, for the greyscale  $f$  there are only two possibilities: either  $f(u) = f(v)$  and  $l$  is odd or else  $f(u) \neq f(v)$  and  $l$  is even. It is straightforwardly checked that there is a vertex  $w \in P_{uv}$  such that  $f(w) \notin \{0, 1\}$ , since otherwise, the first component of the maximum contrast vector  $\mathcal{C}_f$  is 0.  $\square$

Next result gives a bound set of  $Im(f)$  for the case of 2-chromatic connected graphs with  $V_{\phi_0}$  or  $V_{\phi_1}$  having precisely one vertex.

**Theorem 5.6.** *Let  $f$  be a maximum contrast greyscale for the  $\{0, 1\}$ -RMACG problem on a 2-chromatic connected graph  $G$ , with  $|V_{\phi_0}| = 1$  or  $|V_{\phi_1}| = 1$ . Then, it is verified  $Im(f) \subseteq \{0, \frac{1}{2}, 1\}$ .*

*Proof.* Let  $f$  be a maximum contrast greyscale compatible with an incomplete  $V_c$ -greyscale  $g$  of  $G$ . Without loss of generality, we deal with the case  $V_{\phi_0} \neq \emptyset$  and  $V_{\phi_1} \neq \emptyset$ , since otherwise  $\text{Im}(f) = \{0, 1\}$ . Besides, we can suppose  $V_{\phi_0} = \{v_0\}$  and the case  $V_{\phi_1} = \{v_0\}$  is analogous.

From Lemma 5.5,  $f$  takes at least a grey tone different from 0 and 1, and from Proposition 5.2, the first component of the vector  $\mathcal{C}_f$  is at most  $\frac{1}{2}$ .

Let us define the following greyscale:

$$f_0(v) = \begin{cases} g(v) & \text{if } v \in V_c \\ \frac{1}{2} & \text{if } v \in N(v_0) \\ \phi_1(v) & \text{otherwise.} \end{cases}$$

It is checked that  $f_0$  is well defined. In particular, no pair of adjacent vertices are coloured with the same grey tone since there are no triangles in the subgraph induced by  $N(v_0)$  and  $\phi_1$  is a proper colouring on  $V - N(v_0)$ . Hence, the first component of the vector  $\mathcal{C}_{f_0}$  is  $\frac{1}{2}$ . Therefore, since  $\mathcal{C}_f \geq \mathcal{C}_{f_0}$ , then  $\mathcal{C}_f$  has its first component equal to  $\frac{1}{2}$ . Finally, from Proposition 5.2 we reach  $\text{Im}(f) = \{0, \frac{1}{2}, 1\}$ .  $\square$

**Proposition 5.7.** *Let  $T$  be a subdivision of the star graph  $K_{1,n}$  with leaves  $\{v_1, \dots, v_n\}$ , for  $n \geq 3$ ,  $u$  the vertex of degree  $n$  in  $T$  and  $g$  an incomplete greyscale such that  $g(v_i) \in \{0, 1\}$ , for  $1 \leq i \leq n$ . For any maximum contrast greyscale compatible with  $g$ ,  $f$  of  $T$ , it is verified that  $\text{Im}(f) \subseteq \{0, \frac{1}{2}, 1\}$ . Moreover,  $f$  uses precisely the grey tone  $\frac{1}{2}$  over at most  $\lfloor \frac{n}{2} \rfloor$  vertices, each one of them lying in a different path  $P_{uv_i}$ .*

*Proof.* Firstly, we observe that in case that  $T$  is equal to  $K_{1,n}$  and  $g(v_i) = 0$  for  $1 \leq i \leq n$  or  $g(v_i) = 1$  for  $1 \leq i \leq n$ , the grey tone  $\frac{1}{2}$  is not used in  $f$ , and hence the result is held. Next, let us observe that since the paths  $P_{uv_i}$  intersect only in the vertex  $u$ , we can define a greyscale  $f'$  that assigns  $f'(u) = \frac{1}{2}$ ,  $f'(v_i) = g(v_i)$  and extends this colouring for each  $P_{uv_i}$  by starting at  $v_i$  and alternates 0 and 1 until the vertices of  $N(u)$  are reached.  $\mathcal{C}_{f'}$  has precisely its  $n$  first components equal to  $\frac{1}{2}$  and the rest of components are equal to 1. Hence the first component of the maximum contrast vector for the  $\{0, 1\}$ -RMACG problem is greater than or equal to  $\frac{1}{2}$  and therefore, by applying Proposition 5.2, any maximum contrast greyscale compatible with  $g$ ,  $f$ , verifies  $\text{Im}(f) \subseteq \{0, \frac{1}{2}, 1\}$ .

Moreover, in case that  $f(u) = \frac{1}{2}$ , then  $\mathcal{C}_{f'} = \mathcal{C}_f$ , hence  $u$  is the only vertex with grey tone equal to  $\frac{1}{2}$ . We reach to the assertion of the Lemma since  $\lfloor \frac{n}{2} \rfloor \geq 1$ .

Now, let us consider the case in which  $f(u) \neq \frac{1}{2}$ . Without loss of generality we set the colouring  $\phi_0$  to be the one that assigns the colour 0 to the vertex  $u$  and denote by  $n_0$  the number of paths  $P_{uv_i}$  verifying  $g(v_i) = \phi_0(v_i)$ . Next, the colouring  $\phi_1$  assigns the colour 1 to the vertex  $u$  and let us denote by  $n_1$  the number of paths  $P_{uv_i}$  verifying  $g(v_i) = \phi_1(v_i)$ .



Next, we select the colour  $c$  for  $u$  such that  $n_c = \max\{n_0, n_1\}$ . By the classical pigeonhole principle, it is deduced that  $n_c \geq \lceil \frac{n}{2} \rceil$ .

Finally, let us define  $f''$  a greyscale compatible with  $g$  in the following way:

$$f''(v) = \begin{cases} g(v) & \text{if } v = v_i \text{ for } i = 1, \dots, n \\ \frac{1}{2} & \text{if } v \in N(v_i) \text{ for each vertex } v_i \text{ such that } \phi_c(v_i) \neq g(v_i) \\ \phi_c(v) & \text{in other case.} \end{cases}$$

This way, only a vertex of  $P_{uv_i}$  has grey tone  $\frac{1}{2}$ , for at most  $n - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor$  paths  $P_{uv_i}$ . We conclude that since the maximum contrast greyscale compatible with  $g$ ,  $f$  of  $G$ , verifies  $\mathcal{C}_f \geq \mathcal{C}_{f''}$ , the assertion holds.  $\square$

The next result shows another case in which the grey tone  $\frac{1}{2}$  plays an important role in the maximum contrast greyscale for the  $\{0, 1\}$ -RMACG problem. It is tackled in the family of trees with precisely three initially coloured vertices and a method to assign the value  $\frac{1}{2}$  is described into the proof.

**Theorem 5.8.** *Any maximum contrast greyscale  $f$  for the  $\{0, 1\}$ -RMACG problem on a tree  $T = G(V, E)$  with an incomplete  $V_c$ -greyscale  $g$  and  $|V_c| = 3$  verifies that  $|\{v \in V : f(v) = \frac{1}{2}\}| \leq 2$ .*

*Proof.* Let  $V_c = \{v_1, v_2, v_3\} \subset V$  and let us consider the 2-colourings  $\phi_0$  and  $\phi_1$  on  $T$ . Since  $T$  is connected, the set  $V$  is partitioned into two chromatic classes in a unique way. Let us select the 2-colouring  $\phi_i$  such that the cardinality of the set  $V_{\phi_i}$  is greater than or equal to 2. Let us suppose  $|V_{\phi_0}| \geq 2$  and the other case is analogous.

If  $|V_{\phi_0}| = 3$ , the greyscale  $f$  defined by  $f(v) = \phi_0(v)$ , for any  $v \in V$ , is a maximum contrast greyscale for the  $\{0, 1\}$ -RMACG problem and there is no vertex with grey tone  $\frac{1}{2}$ , hence the result holds.

Otherwise,  $|V_{\phi_0}| = 2$ , hence  $|V_{\phi_1}| = 1$  and then, from Theorem 5.6, we get  $Im(f) = \{0, \frac{1}{2}, 1\}$ . Moreover, the number of components equal to  $\frac{1}{2}$  in the maximum contrast vector  $\mathcal{C}_f$  coincides with the number of vertices adjacent to those vertices coloured with the grey tone  $\frac{1}{2}$ , that is the sum of the degrees of all vertices coloured with  $\frac{1}{2}$ . Without loss of generality, let us suppose  $\phi_0(v_1) = g(v_1)$ ,  $\phi_0(v_2) = g(v_2)$  and  $\phi_0(v_3) \neq g(v_3)$ . By Lemma 5.5, there is a vertex  $w_1 \in P_{v_1v_3}$  and a vertex  $w_2 \in P_{v_2v_3}$  with  $f(w_1) = f(w_2) = \frac{1}{2}$ . Since  $f$  is a maximum contrast greyscale compatible with  $g$  on  $T$ , the restriction of  $f$  to the union of the three paths  $P_{v_i v_j}$ , for  $1 \leq i < j \leq 3$ , assigns the grey tone  $\frac{1}{2}$  only to  $w_1$  and  $w_2$ . Due to the fact that  $T$  has no cycles, it is straightforwardly checked that, by starting from each  $v_i \in V_c$ ,  $f$  assigns 0 and 1

appropriately to the remaining vertices of  $T - \{V_c \cup \{w_1, w_2\}\}$ . Hence,  $\{w_1, w_2\}$  are the only vertices with grey tone  $\frac{1}{2}$ .

More precisely, let  $W$  be the set of vertices within  $V(P_{v_1v_3} \cup P_{v_2v_3} \cup P_{v_1v_2}) - V_c$  with minimum degree in  $T$ . If there exists a vertex  $w \in W$  such that  $w \in V(P_{v_1v_3}) \cap V(P_{v_2v_3})$ , then  $w_1 = w_2 = w$  and  $f$  assigns the grey tone  $\frac{1}{2}$  to precisely one of such a vertex  $w$ . Otherwise, without loss of generality, let us suppose  $w \in V(P_{v_1v_3}) - V(P_{v_2v_3})$  and consider a pair of vertices  $w' \in V(P_{v_1v_3}) \cap V(P_{v_2v_3})$  and  $w'' \in V(P_{v_2v_3}) - V(P_{v_1v_3})$  both with minimum degree in  $T$ . If  $\deg(w') \leq \deg(w) + \deg(w'')$ , then  $w_1 = w_2 = w'$  and  $f$  assigns the grey tone  $\frac{1}{2}$  to precisely one of such a vertex  $w'$ . If not,  $f$  assigns  $\frac{1}{2}$  to precisely  $w_1 = w$  and  $w_2 = w''$  or to another pair of vertices  $w$  and  $w''$  verifying the same conditions as  $w_1$  and  $w_2$ , respectively.  $\square$

We guess that finding the solution of the  $\{0, 1\}$ -RMACG problem on bipartite graphs is NP-complete, in general. In fact, we have given several examples showing that there are trees with few vertices that need precisely the full set of 5 grey tones given by Theorem 5.3 (see Figure 4). Many other examples of trees with similar properties may be found. Hence, finding the solution of the  $\{0, 1\}$ -RMACG problem is not an easy question to answer even for the case of trees.

## 6 Results and open questions

We have introduced the new concept of contrast of a graph related to vertex and edge colourings. The maximum contrast problem (MACG problem) has been studied in the general case and links with the chromatic number of a graph is presented in Theorems 3.4 and 3.5. As a conclusion, we get that the proposed problem belongs to NP-complete problems category (Theorem 3.6).

However, we have achieved several results that allow us to compute the set of all possible values of grey tones for maximum contrast greyscales of graphs with known chromatic number. Moreover, some notions of subsets in  $[0, 1]$  are introduced, such as the  $h$ -minimum-step-enchained set. We want to remark here that this kind of sets may be considered independently from any graph and could be useful in other branches of Mathematics. By using the algorithmic procedure MAXIMAL ENCHAINED SET included in this paper we can prove that the set of possible values of grey tones of a maximum contrast greyscale of a graph is a rational finite set (Theorems 4.3 and 4.4).

Hence, some natural questions remain open: to design heuristics to approximate the solution of the MACG problem on graphs and for tackling this problem in particular families of graphs in a more efficient way.

On the other hand, we give another version of the MACG problem consisting of solving the same question but initially assigning some extreme tones to a subset of vertices. We name it  $\{0, 1\}$ -Restricted maximum contrast on graphs,  $\{0, 1\}$ -RMACG problem in short. We have provided a bound set of values of a solution for the  $\{0, 1\}$ -RMACG problem on any bipartite graph (Theorem 5.3). The particular case of the  $\{0, 1\}$ -RMACG problem on bipartite complete graphs is solved (Theorem 5.4). We guess that the  $\{0, 1\}$ -RMACG problem on any bipartite graph is NP-complete. Nevertheless, we have solved this problem for bipartite graphs and for particular family of trees with additional conditions on the set of vertices initially coloured (Theorems 5.6 and 5.8). Hence, another improvement of these results like solving the problem in other families of graphs such that outerplanar graphs or planar graphs would be dealt in future works. Besides, another open problem appears if we change the restriction of initially coloured vertices with extreme tones  $\{0, 1\}$  and consider any other grey tones for the incomplete greyscale.

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